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RESEARCH ARTICLE

Poisson structures on basic cycles

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Abstract The Poisson structures on a basic cycle are determined completely via quiver techniques. As a consequence, all Poisson structures on basic cycles are inner.

Keywords Poisson algebra, inner Poisson structure, basic cycle
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1 Introduction

The Poisson algebras appear naturally in the Poisson geometry. The Poisson manifold is a differential manifold M such that the algebra of smooth functions over M is equipped with a bilinear map called the Poisson bracket, turning it into a Poisson algebra. With the development of noncommutative geometry, noncommutative versions of Poisson algebras have been introduced and investigated from different perspectives, see, for examples, [3–6,9,10].

In this paper, we follow the notion of noncommutative Poisson algebras as introduced in [5]. By definition, a *Poisson algebra* over a field $\mathbb{K}$ means a triple $(A, \cdot, \{\text{-},\text{-}\})$, where $(A, \cdot)$ is an associative $\mathbb{K}$-algebra and $(A, \{\text{-},\text{-}\})$ is a Lie algebra over $\mathbb{K}$, such that the Leibniz rule

$$\{a, bc\} = \{a, b\}c + b\{a, c\},$$

or equivalently,

$$\{ab, c\} = \{a, c\}b + a\{b, c\}$$

holds for all $a, b, c \in A$. We also call $(A, \cdot, \{\text{-},\text{-}\})$ a Poisson structure on the algebra $(A, \cdot)$ or on the Lie algebra $(A, \{\text{-},\text{-}\})$. This version of Poisson algebras has been widely investigated by many mathematicians recently, for examples, [7,8,11]. Notice that as an associative algebra, A is not required to be commutative.

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There are two natural ways to construct the Poisson algebras, say from given associative algebras [11] and from given Lie algebras [7]. Clearly, any noncommutative $\mathbb{K}$-algebra $(A,\cdot)$ has a natural standard Poisson structure $(A,\cdot,\lambda[\text{-},\text{-}])$, where $[\text{-},\text{-}]$ is the commutator, i.e.,

$$[a,b]=ab-ba,\quad \forall\ a,b\in A,$$

and $\lambda\in\mathbb{K}$.

In fact, all Poisson structures on many known noncommutative associative algebras are standard, for example, on the simple algebras [7], on the algebras of upper triangular matrices $T_n(\mathbb{K})$ [7], on the poset subalgebras of $M_\infty(\mathbb{C})$ [8], and on the noncommutative prime algebras [6]. It seems that the first example of noncommutative, non-standard Poisson algebras were obtained in [11] via the quiver technique by considering the so-called inner and outer Poisson structures on path algebras.

In this paper, we determine the Poisson structures on basic cycles completely. In Section 2, we classify all Poisson structures on the path algebra of a basic cycle. Section 3 deals with the path algebra of a basic cycle with any relation. As we show, all Poisson structures on basic cycles are inner in the sense of [11].

Throughout the paper, $\mathbb{K}$ is a field of characteristic 0. For unexplained notations on quivers, we refer to [1,2].

2 Poisson structures on a basic cycle

Let $(A,\cdot,\{\text{-},\text{-}\})$ be a Poisson algebra. For $a\in A$, the linear transformation $\{a,\text{-}\}\colon A\to A$ is called the Hamiltonian of a and denoted by $\mathrm{ham}(a)$. The Leibniz rule just says that $\mathrm{ham}(a)$ is a derivation of the associative algebra $(A,\cdot)$. A Poisson algebra $(A,\cdot,\{\text{-},\text{-}\})$ is said to be inner if $\mathrm{ham}(a)$ is an inner derivation of $(A,\cdot)$ (i.e., $\mathrm{ham}(a)=[a',\text{-}]$ for some $a'\in A$) for all $a\in A$.

Recall a characterization of inner Poisson structure in [11]. Denote by $\mathscr{P}(A)$ the set of all $\mathbb{K}$-linear transformations g of A satisfying

$$[g(x),y]=[x,g(y)],\tag{2.1}$$

$$[g(x),g(y)]-g([g(x),y])\in Z(A)\tag{2.2}$$

for any $x,y\in A$, and

$$Z(A)\subseteq \operatorname{Ker} g.\tag{2.3}$$

Lemma 2.1 [11] *Let $(A,\cdot)$ be an associative algebra. Let $(A,\cdot,\{\text{-},\text{-}\})$ be an inner Poisson algebra on $(A,\cdot)$. Then there exists some $\mathbb{K}$-linear transformation g of A satisfying* (2.1)–(2.3) *such that*

$$\mathrm{ham}(a)=[g(a),\text{-}],\quad \forall\ a\in A.$$

Conversely, let g be a $\mathbb{K}$-linear transformation of A satisfying (2.1) *and* (2.2). *Then g induces an inner Poisson structure $(A, \cdot, \{\text{-},\text{-}\})$ on A, whose Lie bracket is given by*

$$\{a, b\} = [g(a), b], \quad \forall\ a, b \in A.$$

We also need the following lemma for later use.

Lemma 2.2 [11] *Let $(A, \cdot, \{\text{-},\text{-}\})$ be a Poisson algebra, and let $\{e_1, \ldots, e_n\}$ be a set of pairwise orthogonal idempotents of $(A, \cdot)$. Then*

(i) $\{e_i, e_j\} = 0$ *for any* $1 \leqslant i, j \leqslant n$.

(ii) *If* $x \in e_iAe_j,\ y \in e_pAe_q$, *and* $j \neq p,\ q \neq i$, *then*

$$\{x, y\} = e_ie_p\{x, y\}e_qe_j.$$

In particular, if in addition $i \neq p$ or $j \neq q$, then

$$\{e_iAe_j, e_pAe_q\} = 0.$$

Recall that a quiver $Q = (Q_0, Q_1, s, t)$ is an oriented graph, where Q_0 is the set of vertices $\{e_1, \ldots, e_n\}$, Q_1 is the set of arrows, and for any arrow α, $s(\alpha)$ and $t(\alpha)$ are the source and target vertices of α, respectively. Denote by $\mathbb{K}Q$ the path algebra of Q. Let Q be a finite quiver, and let I be an admissible ideal of $\mathbb{K}Q$. We call the pair (Q, I) a bound quiver and $\mathbb{K}Q/I$ the *bound quiver algebra* of (Q, I), see [1, Definition 2.1]. A famous result of Gabriel says that any finite dimensional elementary algebra can be realized as a bound quiver algebra, see, for instance, [1, Theorem 3.7]. A Poisson algebra $(A, \cdot, \{\text{-},\text{-}\})$ is said to be a *quiver Poisson algebra* of a quiver Q if as an associative algebra, A is either the path algebra or a bound quiver algebra of Q. The following result is easy.

Lemma 2.3 *Let $(A, \cdot, \{\text{-},\text{-}\})$ be a quiver Poisson algebra of a quiver $Q = (Q_0, Q_1)$. Then*

(i) $\{e, \alpha\} = 0$ *if* $e \neq e_{s(\alpha)}, e_{t(\alpha)}$ *for any* $e \in Q_0, \alpha \in Q_1$.

(ii) $\{e_{s(\alpha)}, \alpha\} = \{\alpha, e_{t(\alpha)}\} \in e_{s(\alpha)}Ae_{t(\alpha)}$ *if* $s(\alpha) \neq t(\alpha)$.

Proof (i) follows from Lemma 2.2 (ii) easily, and we need only to prove (ii). Consider $e \in Q_0$ and $\alpha \in Q_1$ with $s(\alpha) \neq t(\alpha)$. By Leibniz rule and Lemma 2.2 (i), we have

$$\{e, \alpha\} = \{e, e_{s(\alpha)}\alpha e_{t(\alpha)}\} = e_{s(\alpha)}\{e, \alpha\}e_{t(\alpha)} \in e_{s(\alpha)}Ae_{t(\alpha)}.$$

Since $s(\alpha) \neq t(\alpha)$, we get

$$0 = \{\alpha, e_{s(\alpha)}e_{t(\alpha)}\} = e_{s(\alpha)}\{\alpha, e_{t(\alpha)}\} + \{\alpha, e_{s(\alpha)}\}e_{t(\alpha)},$$

and hence,

$$\{\alpha, e_{t(\alpha)}\} = \{e_{s(\alpha)}, \alpha\}. \qquad \square$$

Now, we focus on basic cycles. Recall that a basic cycle of length n, denoted by C_n, is an oriented graph with n vertices $e_1, \dots, e_n$ and a unique arrow α_i from e_i to e_{i+1} for each $1 \leqslant i \leqslant n$, here, we take the indices modulo n. Denote by p_i^s the unique path in C_n of length s and starting at e_i, in particular, we have $p_i^0 = e_i$, $p_i^1 = \alpha_i$. Clearly, the path algebra $\mathbb{K}C_n$ has a $\mathbb{K}$-basis $\{p_i^s \mid s \geqslant 0; i = 1, \dots, n\}$ and the multiplication is given by

$$p_i^s p_j^t = \delta_{i+s,j} p_i^{s+t},$$

where $\delta_{i+s,j}$ is Kronecker's delta. Since $\mathbb{K}C_n$ is generated by vertices and arrows, by Leibniz rule, a Poisson bracket on $\mathbb{K}C_n$ is uniquely determined by $\{p_i^s, p_j^t\}$, $1 \leqslant i, j \leqslant n$ and $s, t = 0, 1$.

Remark 2.4 For $n = 1$, the path algebra $\mathbb{K}C_n$ is a polynomial algebra in one variable. It is well known that the only Poisson structure on $\mathbb{K}C_n$ is the trivial one. The reason is that for a Poisson algebra, 1 is always in the center of the Lie algebra.

Lemma 2.5 *Let $(A, \cdot, \{\text{-},\text{-}\})$ be a quiver Poisson algebra of the basic cycle C_n, where $n \geqslant 2$ is a given integer. Then*

(i) $\{\alpha_i, \alpha_j\} = 0$, *if* $i - j \neq \pm 1$;

(ii) $\alpha_{i-1}\{e_i, \alpha_i\} = \{\alpha_{i-1}, e_i\}\alpha_i \in e_{i-1}Ae_{i+1}$.

Proof (i) It is obvious by Lemma 2.2 (ii).

(ii) By Lemma 2.2 (ii) and Leibniz rule, we have

$$0 = \{e_i, \alpha_{i-1}\alpha_i\} = \alpha_{i-1}\{e_i, \alpha_i\} + \{e_i, \alpha_{i-1}\}\alpha_i.$$

Clearly,

$$\alpha_{i-1}\{e_i, \alpha_i\} = \{\alpha_{i-1}, e_i\}\alpha_i \in e_{i-1}Ae_{i+1}. \qquad \square$$

Combined with Lemma 2.3, we obtain the following result, which is crucial in determining Poisson structures on $\mathbb{K}C_n$. Note that by Lemma 2.3,

$$\{e_1, \alpha_1\} = \sum_{0 \leqslant l \leqslant q} \lambda_l p_1^{ln+1}$$

for some integer $q \geqslant 0$ and $\lambda_l \in \mathbb{K}$.

Corollary 2.6 *Let $(\mathbb{K}C_n, \cdot, \{\text{-},\text{-}\})$ be a Poisson algebra, and let*

$$\{e_1, \alpha_1\} = \sum_{0 \leqslant l \leqslant q} \lambda_l p_1^{ln+1},$$

where $q \geqslant 0$ is an integer and $\lambda_l \in \mathbb{K}$. Then

$$\{e_i, \alpha_i\} = \sum_{0 \leqslant l \leqslant q} \lambda_l p_i^{ln+1}, \quad i = 2, \dots, n.$$

Proof By Lemma 2.3 (ii), we may assume that

$$\{e_i, \alpha_i\} = \sum_{0 \leqslant l \leqslant q_i} \lambda_i^{(l)} p_i^{ln+1},$$

where for $i = 1, \ldots, n,\ q_i \geqslant 0$ is an integer determined by i and $\lambda_i^{(l)} \in \mathbb{K}$. Again by Lemmas 2.3 (ii) and 2.5 (ii), we have

$$\alpha_{i-1}\{e_i, \alpha_i\} = \{\alpha_{i-1}, e_i\}\alpha_i = \{e_{i-1}, \alpha_{i-1}\}\alpha_i, \quad i = 1, \ldots, n.$$

Hence,

$$\begin{aligned} \sum_{0 \leqslant l \leqslant q_i} \lambda_i^{(l)} p_{i-1}^{ln+2} &= \alpha_{i-1} \sum_{0 \leqslant l \leqslant q_i} \lambda_i^{(l)} p_i^{ln+1} \\ &= \sum_{0 \leqslant l \leqslant q_{i-1}} \lambda_{i-1}^{(l)} p_{i-1}^{ln+1} \alpha_i \\ &= \sum_{0 \leqslant l \leqslant q_{i-1}} \lambda_{i-1}^{(l)} p_{i-1}^{ln+2}. \end{aligned}$$

It follows that $\lambda_1^{(l)} = \cdots = \lambda_n^{(l)} = \lambda_l$ for any l, and $q_1 = \cdots = q_n = q$. □

Proposition 2.7 *Let $(\mathbb{K}C_n, \cdot, \{\text{-},\text{-}\})$ be a Poisson algebra with $n \geqslant 2$. Then there exists an integer $q \geqslant 0$ and a vector $(\lambda_0, \lambda_1, \ldots, \lambda_q) \in \mathbb{K}^{q+1}$ such that*

$$\{p_i^s, p_j^t\} = \sum_{0 \leqslant l \leqslant q} \lambda_l (p_i^{ln+s} p_j^t - p_j^{ln+t} p_i^s), \quad \forall\ i, j = 1, \ldots, n,\ \forall\ s, t \geqslant 0.$$

Proof By Lemma 2.3 (ii), we may assume that

$$\{e_1, \alpha_1\} = \sum_{0 \leqslant l \leqslant q} \lambda_l p_1^{ln+1}.$$

If $n = 2$, then

$$\{\alpha_1, \alpha_2\} = \{\alpha_1, e_2\}\alpha_2 - \{\alpha_2, e_1\}\alpha_1.$$

If $n \geqslant 3$, then, by Lemma 2.5 and Corollary 2.6, it is easy to check that

$$\{\alpha_i, \alpha_j\} = \begin{cases} \sum\limits_{0 \leqslant l \leqslant q} \lambda_l p_i^{ln+2}, & j = i+1; \\ -\sum\limits_{0 \leqslant l \leqslant q} \lambda_l p_{i-1}^{ln+2}, & j = i-1; \\ 0, & \text{otherwise.} \end{cases}$$

In any case, we have

$$\{\alpha_i, \alpha_j\} = \sum_{0 \leqslant l \leqslant q} \lambda_l (p_i^{ln+1}\alpha_j - p_j^{ln+1}\alpha_i), \quad \forall\ i, j = 1, \ldots, n.$$

By the Leibniz rule, we have

$$
\begin{aligned}
\{p_i^s, p_j^t\} &= \{\alpha_i\alpha_{i+1}\cdots\alpha_{i+s-1}, \alpha_j\alpha_{j+1}\cdots\alpha_{j+t-1}\} \\
&= \sum_{i\leqslant l\leqslant i+s-1} \alpha_i\cdots\alpha_{l-1}\{\alpha_l, \alpha_j\alpha_{j+1}\cdots\alpha_{j+t-1}\}\alpha_{l+1}\cdots\alpha_{i+s-1} \\
&= \sum_{\substack{i\leqslant l\leqslant i+s-1 \\ j\leqslant k\leqslant j+t-1}} \alpha_i\cdots\alpha_{l-1}\alpha_j\cdots\alpha_{k-1}\{\alpha_l,\alpha_k\}\alpha_{k+1}\cdots\alpha_{j+t-1}\alpha_{l+1}\cdots\alpha_{i+s-1} \\
&= \alpha_i\cdots\alpha_{i+s-2}\{\alpha_{i+s-1},\alpha_j\}\alpha_{j+1}\cdots\alpha_{j+t-1} \\
&\quad - \alpha_j\cdots\alpha_{j+t-2}\{\alpha_{j+t-1},\alpha_i\}\alpha_{i+1}\cdots\alpha_{i+s-1} \\
&= p_i^{s-1}\Big(\sum_{0\leqslant l\leqslant q}\lambda_l p_{i+s-1}^{ln+1}\Big)p_j^t - p_j^{t-1}\Big(\sum_{0\leqslant l\leqslant q}\lambda_l p_{j+t-1}^{ln+1}\Big)p_i^s \\
&= \sum_{0\leqslant l\leqslant q}\lambda_l(p_i^{ln+s}p_j^t - p_j^{ln+t}p_i^s). \qquad \square
\end{aligned}
$$

Now, applying Proposition 2.7, we obtain all Poisson structures on the basic cycle C_n, which gives the main result of this section.

Theorem 2.8 *Let C_n be a basic cycle with $n \geqslant 2$, and let $\mathbb{K}C_n$ be the path algebra. Then, for any Poisson structure $(\mathbb{K}C_n, \cdot, \{\text{-},\text{-}\})$ on $\mathbb{K}C_n$, there exists a $\mathbb{K}$-linear transformation $g\colon \mathbb{K}C_n \to \mathbb{K}C_n$ satisfying (2.1) and (2.2) such that*

$$\mathrm{ham}(x) = [g(x), \text{-}], \quad \forall\, x \in \mathbb{K}C_n.$$

Consequently, any Poisson structure on a basic cycle is inner.

Conversely, for any integer $q \geqslant 0$ and any vector $(\lambda_0, \lambda_1, \ldots, \lambda_q) \in \mathbb{K}^{q+1}$, the $\mathbb{K}$-linear transformation

$$
\begin{aligned}
g\colon\ \mathbb{K}C_n &\to \mathbb{K}C_n, \\
p_i^s &\mapsto \sum_{0\leqslant l\leqslant q}\lambda_l p_i^{ln+s}
\end{aligned}
$$

induces a Poisson structure on $\mathbb{K}C_n$, and the Lie bracket is given by

$$\{p_i^s, p_j^t\} = \sum_{0\leqslant l\leqslant q}\lambda_l(p_i^{ln+s}p_j^t - p_j^{ln+t}p_i^s), \quad \forall\, i,j = 1,\ldots,n,\ \forall\, s,t \geqslant 0.$$

Proof Let $(\mathbb{K}C_n, \cdot, \{\text{-},\text{-}\})$ be a given Poisson algebra. We may define a $\mathbb{K}$-linear transformation $g\colon \mathbb{K}C_n \to \mathbb{K}C_n$ by setting

$$g(p_i^s) = \sum_{0\leqslant l\leqslant q}\lambda_l p_i^{ln+s}$$

for any $1 \leqslant i \leqslant n$ and $s \geqslant 0$, here q and λ_l's are determined by $(\mathbb{K}C_n, \cdot, \{\text{-},\text{-}\})$ as given in Proposition 2.7. Then

$$
\begin{aligned}
[g(p_i^s), p_j^t] &= \Big[\sum_{0 \leqslant l \leqslant q} \lambda_l p_i^{ln+s}, p_j^t \Big] \\
&= \sum_{0 \leqslant l \leqslant q} \lambda_l (p_i^{ln+s} p_j^t - p_j^t p_i^{ln+s}) \\
&= \sum_{0 \leqslant l \leqslant q} \lambda_l (p_i^{ln+s} p_j^t - p_j^{ln+t} p_i^s) \\
&= \{p_i^s, p_j^t\}, \quad \forall\, i,j = 1, \dots, n, \ \forall\, s,t \geqslant 0.
\end{aligned}
$$

It follows that

$$\{x, \text{-}\} = [g(x), \text{-}], \quad \forall\, x \in \mathbb{K}C_n,$$

and hence, $(\mathbb{K}C_n, \cdot, \{\text{-},\text{-}\})$ is inner.

For the converse part, it suffices to show that g satisfies (2.1) and (2.2). In fact, by the definition of g, we have

$$
\left.
\begin{aligned}
[g(p_i^s), p_j^t] &= [p_i^s, g(p_j^t)], \\
[g(p_i^s), g(p_j^t)] &= g[g(p_i^s), p_j^t],
\end{aligned}
\right. \quad \forall\, i,j = 1, \dots, n, \ \forall\, s,t \geqslant 0.
$$

By Lemma 2.1, we know that g gives a Poisson structure on $\mathbb{K}C_n$, and the bracket is clear. □

3 Poisson structures on a basic cycle with relations

Let $A = \mathbb{K}C_n/I$ be the bound quiver algebra of the basic cycle C_n $(n \geqslant 2)$ with an admissible ideal I. One can easily show that an admissible ideal must be monomial for a basic cycle. Moreover, for any vertex e_i, there exists a maximal l_i such that $p_i^{l_i}$ is not in I. Such an l_i always exists since I is admissible and $p_i^l \in I$ for sufficiently large l. Clearly, I is generated by the paths $p_i^{l_i+1}, 1 \leqslant i \leqslant n$. Without loss of generality, we may assume that

$$l_1 = \min_{1 \leqslant i \leqslant n} l_i, \quad l_i - 1 = nq_i + r_i, \quad q_i \in \mathbb{N}, \ 0 \leqslant r_i \leqslant n-1, \ 1 \leqslant i \leqslant n.$$

By definition, we have the following property.

Proposition 3.1 *Let A, I, l_i, q_i, and r_i be as above, and assume that $l_1 = \min_{1 \leqslant i \leqslant n} l_i$. Then $q_1 \leqslant q_i \leqslant q_1 + 1$. Moreover, $r_i \geqslant r_1$ if $q_i = q_1$, and $r_i \leqslant r_1 - i + 1$ if $q_i = q_1 + 1$.*

Proof Since $p_i^{l_i}$ is the longest non-zero path in A with the source i, we have $l_i \leqslant l_{i+1} + 1$, and hence, $l_1 \leqslant l_i \leqslant l_1 + n - i + 1$ by the minimality of l_1, or equivalently,

$$q_1 + \frac{r_1 - r_i}{n} \leqslant q_i \leqslant q_1 + 1 + \frac{r_1 - r_i - i + 1}{n}. \tag{3.1}$$

Clearly,

$$-(n-1) \leqslant r_1 - r_i \leqslant n - 1$$

for each $0 \leqslant r_i \leqslant n-1$. Therefore, we have $q_1 \leqslant q_i \leqslant q_1+1$. The rest part follows easily from inequality (3.1). $\square$

Similar to the path algebra case, a Poisson structure on $\mathbb{K}C_n/I$ is uniquely determined by $\{p_i^s, p_j^t\},\ s,t=0,1,\ 1\leqslant i,j\leqslant n$. Let E denote the set of arrows $\{\alpha_i \mid r_i=0\}$. Set $V=\{e_i \mid q_i=q_1\}$ if $r_1\neq 0$, and $V=\emptyset$ if $r_1=0$ for consistency. Remove the arrows in E and the vertices in V from the quiver C_n, and assume that the remained subquiver has m connected components, say $P_1, P_2, \ldots, P_m$. Notice that when a vertex is removed from a quiver, all arrows incident to it will be removed automatically. Clearly, we have the following fact.

Lemma 3.2 *Let $A,\ I,\ l_i,\ q_i,\ r_i\ (1\leqslant i\leqslant n)$, and $P_k\ (1\leqslant k\leqslant m)$ be as above.*

(i) *If $r_1=0$ and $p_i^{nq_1+r}\neq 0$ for some $r\geqslant 2$, then $e_i, e_{i+1}, \ldots, e_{i+r-1}$ belong to the same component P_k for some k.*

(ii) *If $r_1\neq 0$ and $p_i^{n(q_1+1)+r}\neq 0$ for some $r\geqslant 2$, then $e_i, e_{i+1}, \ldots, e_{i+r-1}$ belong to the same component P_k for some k.*

Proof If $r_1=0$ and $p_i^{nq_1+r}\neq 0,\ r\geqslant 2$, then $p_k^{nq_1+2}\neq 0$ for $i\leqslant k<i+r-1$, and hence, $\alpha_k\notin E$. Therefore, $e_i, e_{i+1}, \ldots, e_{j+r-1}$ belong to the same component P_k for some k.

The case $r_1\neq 0$ is similarly proved. $\square$

By Lemmas 2.3 and 2.5, we may assume that

$$\{e_i, \alpha_i\} = \sum_{0\leqslant l\leqslant q_i} \lambda_i^{(l)} p_i^{ln+1}$$

for some $\lambda_i^{(l)}\in\mathbb{K}$. By using the similar argument, we obtain the following analog of Corollary 2.6.

Lemma 3.3 *Let $A,\ I,\ l_i,\ q_i,\ r_i\ (1\leqslant i\leqslant n)$, and $P_k\ (1\leqslant k\leqslant m)$ be as above. Let $(\mathbb{K}C_n/I, \cdot, \{\text{-},\text{-}\})$ be a Poisson algebra, and for each i, let*

$$\{e_i, \alpha_i\} = \sum_{0\leqslant l\leqslant q_i} \lambda_i^{(l)} p_i^{ln+1}.$$

(i) *If $r_1=0$, then $\lambda_1^{(l)}=\cdots=\lambda_n^{(l)}$ for $0\leqslant l\leqslant q_1-1$, and $\lambda_i^{(q_1)}=\lambda_j^{(q_1)}$ when e_i and e_j are in the same component P_k for some k.*

(ii) *If $r_1\neq 0$, then $\lambda_1^{(l)}=\cdots=\lambda_n^{(l)}$ for $0\leqslant l\leqslant q_1$, and $\lambda_i^{(q_1+1)}=\lambda_j^{(q_1+1)}$ when e_i and e_j are in the same component P_k for some k.*

Proof By Lemmas 2.3 (ii) and 2.5 (ii), we have

$$\alpha_{i-1}\{e_i, \alpha_i\} = \{\alpha_{i-1}, e_i\}\alpha_i = \{e_{i-1}, \alpha_{i-1}\}\alpha_i,$$

and hence,

$$\begin{aligned}\sum_{0\leqslant l\leqslant q_i}\lambda_i^{(l)}p_{i-1}^{ln+2} &= \alpha_{i-1}\sum_{0\leqslant l\leqslant q_i}\lambda_i^{(l)}p_i^{ln+1}\\ &= \sum_{0\leqslant l\leqslant q_{i-1}}\lambda_{i-1}^{(l)}p_{i-1}^{ln+1}\alpha_i\\ &= \sum_{0\leqslant l\leqslant q_{i-1}}\lambda_{i-1}^{(l)}p_{i-1}^{ln+2}.\end{aligned}$$

If $r_1 = 0$, then $q_k = q_1$ for all k, and $p_i^{ln+2} \neq 0$ for $0 \leqslant l \leqslant q_1 - 1$ by Proposition 3.1. By comparing the coefficients in the above equality, we have $\lambda_{i-1}^{(l)} = \lambda_i^{(l)}$ for all $0 \leqslant l \leqslant q_1 - 1$ and $1 \leqslant i \leqslant n$. Now, consider the case $l = q_1$. Suppose $i, i-1 \in P_k$ for some $1 \leqslant k \leqslant m$. By definition of P_k, we know that

$$l_{i-1} = nq_1 + r_{i-1} + 1 \geqslant nq_1 + 2$$

since $r_{i-1} \geqslant 1$, and hence, $p_{i-1}^{nq_1+2} \neq 0$. By comparing the coefficients again, we have $\lambda_i^{(q_1)} = \lambda_{i-1}^{(q_1)}$. Since each P_k is a connected subquiver of C_n, vertices in P_k are connected by a sequence of arrows in C_n, which implies that $\lambda_i^{(q_1)} = \lambda_j^{(q_1)}$ for any $i, j \in P_k$.

The case $r_1 \neq 0$ is proved similarly, and we omit it here. □

Theorem 3.4 *Under the same assumption as in Lemma* 3.2, *let* $(\mathbb{K}C_n/I, \cdot, \{\text{-},\text{-}\})$ *be a Poisson algebra.*

(i) *If* $r_1 = 0$, *then there exists a vector* $(\lambda_0, \dots, \lambda_{q_1-1}, \mu_1, \dots, \mu_m) \in \mathbb{K}^{q_1+m}$ *such that*

$$\{p_i^s, p_j^t\} = \sum_{0\leqslant l\leqslant q_{i+s}}\lambda_{i+s}^{(l)}p_i^{ln+t}p_j^s - \sum_{0\leqslant l\leqslant q_{j+t}}\lambda_{j+t}^{(l)}p_j^{ln+s}p_i^t, \tag{3.2}$$

where $\lambda_i^{(l)} = \lambda_l$ *for all* $l = 0, 1, \dots, q_1 - 1$, *and* $\lambda_i^{(q_1)} = \mu_k$ *for all* $i \in P_k$, $k = 1, \dots, m$.

(ii) *If* $r_1 \neq 0$, *then there exists a vector* $(\lambda_0, \dots, \lambda_{q_1}, \mu_1, \dots, \mu_m) \in \mathbb{K}^{q_1+m+1}$ *such that*

$$\{p_i^s, p_j^t\} = \sum_{0\leqslant l\leqslant q_{i+s}}\lambda_{i+s}^{(l)}p_i^{ln+s}p_j^t - \sum_{0\leqslant l\leqslant q_{j+t}}\lambda_{j+t}^{(l)}p_j^{ln+s}p_i^t, \tag{3.3}$$

where $\lambda_i^{(l)} = \lambda_l$ *for all* $l = 0, 1, \dots, q_1$, *and* $\lambda_i^{q_1+1} = \mu_k$ *for all* $i \in P_k$, $k = 1, \dots, m$.

Proof We only prove (i), and the proof of (ii) is similar. By Lemma 3.2, we may assume that

$$\{e_i, \alpha_i\} = \sum_{1\leqslant l\leqslant q_i}\lambda_i^{(l)}p_i^{ln+1},$$

where $\lambda_i^{(l)} = \lambda_l$ for $0 \leqslant l \leqslant q_1 - 1$, and $\lambda_i^{(q_1)} = \mu_k$ for $i \in P_k,\ k = 1, \ldots, m$. By the similar argument in the proof of Theorem 2.8, we obtain equality (3.2), which completes the proof. □

By Theorem 3.4, we conclude that all Poisson structures on $\mathbb{K}C_n/I$ are inner.

Theorem 3.5 *Let $A = \mathbb{K}C_n/I$ be a bound quiver algebra of C_n. Then for any Poisson structure $(A, \cdot, \{\text{-},\text{-}\})$ on A, there exists a $\mathbb{K}$-linear transformation $g\colon A \to A$ satisfying* (2.1) *and* (2.2) *such that*

$$\mathrm{ham}(x) = [g(x), \text{-}], \quad \forall\ x \in A.$$

Consequently, any Poisson structure on a basic cycle is inner.

Conversely, let $l_i,\ q_i,\ r_i$ $(1 \leqslant i \leqslant n)$, and P_k $(1 \leqslant k \leqslant m)$ be as above.

(i) *If $r_1 = 0$, then for any vector $(\lambda_0, \ldots, \lambda_{q_1-1}, \mu_1, \ldots, \mu_m) \in \mathbb{K}^{q_1+m}$, there exists a Poisson structure on $\mathbb{K}C_n/I$ such that the Lie bracket is given by* (3.2).

(ii) *If $r_1 \neq 0$, then for any vector $(\lambda_0, \ldots, \lambda_{q_1}, \mu_1, \ldots, \mu_m) \in \mathbb{K}^{q_1+m+1}$, there exists a Poisson structure on $\mathbb{K}C_n/I$ such that the Lie bracket is given by* (3.3).

Proof Using the same assumption in Lemma 3.3, we only prove that it holds if $r_1 = 0$, and the proof of the case $r_1 \neq 0$ is similar. By Theorem 3.4, we know that each Poisson structure on A corresponds to a vector $(\lambda_0, \ldots, \lambda_{q_1-1}, \mu_1, \ldots, \mu_k)$ if $r_1 = 0$. Let $g\colon A \to A$ be the linear transformation given by

$$g(p_i^s) = \sum_{0 \leqslant l \leqslant q_{i+s}} \lambda_{i+s}^{(l)} p_i^{ln+s} \tag{3.4}$$

for $1 \leqslant s \leqslant l_i$, where $\lambda_i^{(l)} = \lambda_l$ for $0 \leqslant l \leqslant q_1 - 1$; $\lambda_i^{(q_1)} = \mu_k$ for all $i \in P_k,\ k = 1, \ldots, m$; and

$$g(e_i) = -\Big(\sum_{0 \leqslant l \leqslant q_{i-1}} \lambda_{i-1}^{(l)} p_{i-1}^{ln} + \sum_{0 \leqslant l \leqslant q_i} \lambda_i^{(l)} p_{i+1}^{ln} \Big), \tag{3.5}$$

where $q_i' = \max\{q_i, q_{i-1}\}$ for $1 \leqslant i \leqslant n$.

It is direct to check that

$$\{p_i^s, \text{-}\} = [g(p_i^s), \text{-}], \quad \forall\ s = 0, 1, \ldots, l_i,\ \forall\ i = 1, \ldots, n,$$

$$\left.\begin{aligned} [g(p_i^s), p_j^t] &= [p_i^s, g(p_j^t)], \\ [g(p_i^s), g(p_j^t)] &= g([g(p_i^s), p_j^t]), \end{aligned}\right. \quad \forall\ i, j = 1, \ldots, n,\ \forall\ s, t \geqslant 0,$$

by Lemmas 3.2, 3.3, and Theorem 3.4.

Conversely, for any vector $(\lambda_0, \ldots, \lambda_{q_1-1}, \mu_1, \ldots, \mu_m) \in \mathbb{K}^{q_1+m}$, define the $\mathbb{K}$-linear transformation $g\colon A \to A$ given by (3.4) and (3.5), and again one checks that g satisfies (2.1) and (2.2). Therefore, g induces an inner Poisson structure on A. □

Corollary 3.6 *Let $(\mathbb{K}C_n/I, \cdot, \{\text{-},\text{-}\})$ be a Poisson algebra. If $R^n \subset I \subset R^3$, then $\mathbb{K}C_n/I$ is a standard Poisson algebra, where R is the ideal generated by all arrows.*

Proof Clearly, if $R^n \subset I \subset R^3$, then $q_i = 0$ for all $i = 1, 2, \ldots, n$ and $r_i > 0$. By Theorem 3.4 (i), there exists a unique parameter λ_0 such that

$$\{p_i^s, p_j^s\} = \lambda_0[p_i^s, p_j^s], \quad \forall\, i, j = 1, \ldots, n,\ \forall\, s, t \geqslant 0.$$

It follows that $\mathbb{K}C_n/I$ has only standard Poisson structure. □

Example 3.7 Let C_{10} be the basic cycle with 10 vertices, and let I be the admissible ideal generated by p_1^{10}, p_4^{12}, and p_7^{12}. In this case,

$$\begin{aligned} &l_1 = 9, \quad l_2 = 13, \quad l_3 = 12, \quad l_4 = 11, \quad l_5 = 13, \quad l_6 = 12, \\ &l_7 = 11, \quad l_8 = 12, \quad l_9 = 11, \quad l_{10} = 10, \quad q_1 = 0, \quad r_1 = 8. \end{aligned}$$

Clearly, $E = \{\alpha_4, \alpha_7, \alpha_9\}$ and $V = \{e_1, e_{10}\}$ (see Figure 1).

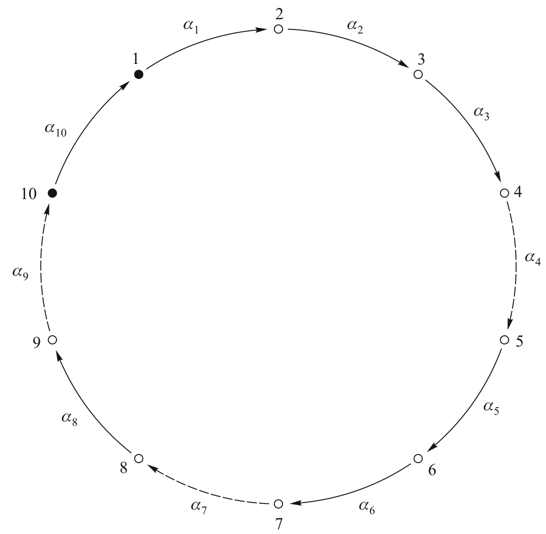

Fig. 1 Basic cycle C_{10} with black vertices in V and dashed arrows in E

Removing the sets E and V from C_{10}, we get three connected components

$$\{2 \to 3 \to 4\}, \quad \{5 \to 6 \to 7\}, \quad \{8 \to 9\}.$$

By Theorem 3.5, we know that any Poisson structure on $\mathbb{K}Q/I$ is determined by a vector $(\lambda_0, \mu_1, \mu_2, \mu_3) \in \mathbb{K}^4$. Conversely, for any given 4-dimensional vector $(\lambda_0, \mu_1, \mu_2, \mu_3)$, we can obtain a Poisson structure on $\mathbb{K}Q/I$ with the Lie bracket induced by

$$\{e_i, \alpha_i\} = \lambda_0\alpha_i + \lambda_i^{(1)} p_i^{11},$$

where

$$\lambda_i^{(1)} = \begin{cases} 0, & i = 1, 10, \\ \mu_1, & i = 2, 3, 4, \\ \mu_2, & i = 5, 6, 7, \\ \mu_3, & i = 8, 9. \end{cases}$$

Example 3.8 Let $A = \mathbb{K}C_n/R^2$, where Q is a basic cycle with n vertices and R is the Jacobson radical of $\mathbb{K}C_n$. By Theorem 3.4, we know that the Poisson structure on A is determined by a vector $(\mu_1, \ldots, \mu_n) \in \mathbb{K}^n$, with the Lie bracket induced by

$$\{e_i, \alpha_i\} = \mu_i \alpha_i, \quad i = 1, \ldots, n.$$

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References

1. Assem I, Simson D, Skowronski A. Elements of the Representations Theory of Associative Algebras. London Math Soc Stud Texts, 65. Cambridge: Cambridge University Press, 2005
2. Auslander M, Reiten I, Smalϕ S O. Representation Theory of Artin Algebras. Cambridge Stud Adv Math, 36. Cambridge: Cambridge University Press, 1995
3. Casasa J M, Datuashvilib T. Noncommutative Leibniz-Poisson algebras. Commun Algebra, 2006, 34: 2507–2530
4. Crawley-Boevey W. Poisson structures on moduli spaces of representations. J Algebra, 2011, 325: 205–215
5. Flato M, Gerstenhaber M, Voronow A A. Cohomology and deformation of Leibniz pairs. Lett Math Phys, 1995, 34: 77–90
6. Farkas D, Letzter G. Ring theory from symplectic geometry. J Pure Appl Algebra, 1998, 125: 155–190
7. Kubo F. Finite-dimensional non-commutative Poisson algebras. J Pure Appl Algebra, 1996, 113: 307–314
8. Kubo F. Non-commutative Poisson algebra structures on affine Kac-Moody algebras. J Pure Appl Algebra, 1998, 126: 267–286
9. Van den Bergh M. Double Poisson algebras. Trans Amer Math Soc, 2008, 360: 5711–5769
10. Xu P. Noncommutative Poisson algebras. Amer J Math, 1994, 116(1): 101–125
11. Yao Y, Ye Y, Zhang P. Quiver Poisson algebras. J Algebra, 2007, 312: 570–589